High-Order Conformal Symplectic FDTD Scheme
Wei Sha\textsuperscript{a} and Xianliang Wu\textsuperscript{b}

\textsuperscript{a} Department of Electrical and Electronic Engineering, the University of Hong Kong, Pokfulam Road, Hong Kong (Email: wsha@eee.hku.hk)
\textsuperscript{b} Key Laboratory of Intelligent Computing \\ & Signal Processing, Anhui University, Feixi Road 3, Hefei 230039, China

The PPT is available at http://www.eee.hku.hk/~wsha/Publications/Files/High-Order\%20Conformal\%20Symplectic\%20FDTD\%20Scheme\_PPT.pdf

1. Introduction
The traditional finite-difference time-domain (FDTD) method [1], which is explicit second-order-accurate in both space and time, has been widely applied to electromagnetic computation and simulation. However, the FDTD method has two primary drawbacks. One is the inability to accurately model curved complex surfaces and material discontinuities by using the staircasing approach with structured grids, and another is the significant accumulated errors from numerical dispersion and anisotropy. Hence fine grids are required to obtain satisfying numerical results, which leads to vast memory requirements and high computational costs, especially for electrically-large domains and for long-term simulation.

For solving the first problem, a variety of conformal techniques have been proposed in [2-5]. For solving the second problem, high-order spatial discretization methods were developed, which can save computational resources by using coarse grids. However, how to model the curved surfaces with the high-order spatial methods is an open question. In other words, can we obtain the satisfying numerical results under both curved boundary and coarse grid conditions?

In order to achieve the goal, the one-sided difference [6] was studied. However, the strategy ignored the shape of objects. Another method is the hybrid subgridding technique [7]. However, it must adopt spatial and temporal interpolation strategies to eliminate the spurious reflections between the coarse grids and the fine grids.

In this paper, a new high-order conformal symplectic FDTD(3,4) scheme is proposed to solve the electromagnetic scattering from three-dimensional curved perfectly conducting objects. The scheme can achieve the above goal without temporal and spatial interpolations.

2. Theory
For the electric field components, the update equations need not to be modified.

\textit{Figure 1}

For the magnetic field components, the general Faraday’s loops are shown in Fig. 1. Using the trick in [8], the semi-discrete update equation for $\mathbf{H}$ field can be written as
\[
\frac{\partial}{\partial t} H_z(i, j + \frac{1}{2}, k + \frac{1}{2}, t) = \frac{9}{8} \frac{1}{\sqrt{\mu_0 \varepsilon_0 \Delta S_i'}} \times \left[ (\hat{E}_z(i, j + \frac{1}{2}, k + 1, t) \hat{l}_{z,i}(i, j, k + 1) - \hat{E}_z(i, j + \frac{1}{2}, k, t) \hat{l}_{z,i}(i, j, k)) \right]

- \left[ (\hat{E}_z(i, j + 1, k + \frac{1}{2}, t) \hat{l}_{z,i}(i, j, k + 1) - \hat{E}_z(i, j, k + \frac{1}{2}, t) \hat{l}_{z,i}(i, j, k)) \right] \right] \]

\[
- \frac{1}{24} \frac{1}{\Delta z} \left( \frac{\hat{E}_y(i, j + \frac{1}{2}, k + 2, t) - \hat{E}_y(i, j + \frac{1}{2}, k - 1, t)}{\Delta z} \right)

\[
- \frac{\hat{E}_y(i, j + 2, k + \frac{1}{2}, t) - \hat{E}_y(i, j - 1, k + \frac{1}{2}, t)}{\Delta y} \right) \] (1)

where \( \Delta_y \) and \( \Delta_z \) are, respectively, the lattice space steps in the \( y \) and \( z \) coordinate directions, \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and the permeability of free space, \( \hat{l}_{x,i} \) and \( \hat{l}_{y,i} \) are the side lengths along the \( y \) and \( z \) directions outside the perfectly conducting object, and \( \Delta S_i' \) is the area of the modified inner loop \( \hat{l}_1 \) outside the object.

To obtain the stable and accurate numerical results, we have

\[
\Delta S_i' = \begin{cases} 
\Delta S_i, \Delta S_i' \geq \gamma \Delta S_i \\
\gamma \Delta S_i, \Delta S_i' < \gamma \Delta S_i 
\end{cases} (2)
\]

where \( \Delta S_i \) is the constant area of the unmodified inner loop \( \hat{l}_i \), and \( \gamma \) is the threshold. The smaller \( \gamma \) results in better numerical precision but smaller time step to ensure the stability of the scheme.

Finally, the three-stage third-order symplectic integrators [9] are employed to discretize the time direction.

Due to the modification for the inner loops of magnetic fields, the decreased CFL number can be estimated as

\[
CFL' = \sqrt{\frac{\lambda_x}{\lambda_y \lambda_z}}, \quad \lambda_y = 2 \sqrt{d} \times \frac{1}{24} + \frac{9}{8}, \quad \lambda_z = \max \left\{ 2 \sqrt{d} \times \frac{1}{24} + \frac{9}{8}, \max \left( \frac{l_{z,i}}{\Delta S_i'} \right) \right\} \quad (3)
\]

where \( d \) is the space dimension, \( \max (l_{z,i}) \) is the maximum length of the four sides connected with the inner loop \( \hat{l}_i \), and \( \lambda_x \) is the temporal stability factor defined in [9], \( \lambda_x = 2 \) and \( \lambda_x = 4.52 \) respectively for the second-order leap-frog method and the third-order symplectic integration scheme.

### 3. Numerical Result

Without loss of generality, it is assumed that a plane wave of frequency 300 MHz travels along the \( z \) direction, the electric field is polarized along the \( x \) direction, and the uniform space step is adopted. The threshold \( \gamma = 0.01 \). For the high-order conformal (HC) SFDTD(3,4) scheme, the CFL number is reduced to 40% of original scheme. For the low-order conformal (LC) FDTD(2,2) method, the CFL number is reduced to 35% of original method.

1. The scattering from electrically-small conducting circular cylinder of height 2m and radius 1m is considered. The symmetrical axis of the cylinder is placed along the \( y \) direction. Fig. 2 shows the relative two-norm errors of bistatic radar cross section (RCS) as a function of points per wavelength (PPW). It can be seen that the HC-SFDTD(3,4) scheme can obtain better
numerical precision than the high-order staircased (HS) SFDTD(3,4) approach and the LC-FDTD(2,2) method.

2. The next example considered is the scattering from electrically-large conducting sphere of diameter $14\text{m}$. In particular, we use only 7 PPW to model the curved surfaces. From Fig. 3, compared with the LC-FDTD(2,2) method and the HS-SFDTD(3,4) approach, the HC-SFDTD(3,4) scheme agrees with the analytical solution very well. The relative two-norm errors of bistatic RCS in H-plane for the LC-FDTD(2,2) method, the HS-SFDTD(3,4) approach, and the HC-SFDTD(3,4) scheme are, respectively, 7.3%, 12.3%, and 0.85%.

Within the same relative two-norm errors bound (1%), we change the settings of the space step and the CFL number, and the CPU time and memory consumed by different algorithms are recorded in Table I. From the table, the HC-SFDTD(3,4) scheme saves considerable memory and CPU time.

4. Conclusion

Using the high-order locally conformal technique and the high-order symplectic integrators, the HC-SFDTD(3,4) scheme is accurate and efficient for modeling the scattering from three-dimensional curved perfectly conducting objects. In addition, the decreased time step caused by the conformal model can be offset by using coarse grids.

References


Fig. 1. The general Faraday’s loops for the fourth-order staggered difference. The dot denotes $H_z$ field. The four electric field components near from $H_z$ link the inner loop $L_1$, and those far from $H_z$ link the outer loop $L_2$. $l_{x}$ and $l_{z}$ are the side lengths along the $y$ and $z$ directions outside the perfectly conducting object. The gray region denotes the region inside the object.

Fig. 2. The relative two-norm errors of bistatic RCS in E-plane.

Fig. 3. The H-plane bistatic RCS of the conducting sphere.

TABLE I  The consumed CPU time and memory under the same relative two-norm errors condition.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>PPW</th>
<th>CFL</th>
<th>Time (s)</th>
<th>Memory (MB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HC-SFDTD(3,4)</td>
<td>7</td>
<td>0.50</td>
<td>5891</td>
<td>258</td>
</tr>
<tr>
<td>HS-SFDTD(3,4)</td>
<td>16</td>
<td>1.00</td>
<td>56279</td>
<td>1318</td>
</tr>
<tr>
<td>LC-FDTD(2,2)</td>
<td>13</td>
<td>0.20</td>
<td>23359</td>
<td>820</td>
</tr>
</tbody>
</table>